Section 3.2

The Derivative as a Function

(1) The Derivative Function

(2) Notation of Derivatives

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2. Constant Multiple Rule
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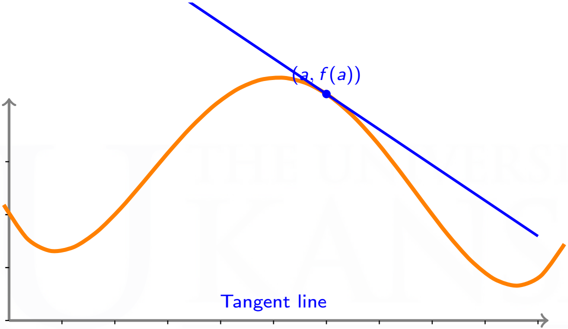
(5) Revisiting Some Examples Using The Rules



Derivative of a Function at a Point

The **derivative** of a function $y = f(x)$ at $x = a$ is (if it exists)

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

For example: The derivative function of $f(x) = x^2 + 4$ is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\left((x+h)^2 + 4\right) - \left(x^2 + 4\right)}{h}$$

$$= \lim_{h\to 0} \frac{\left(x^2 + 2xh + h^2 + 4\right) - \left(x^2 + 4\right)}{h} = \lim_{h\to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h\to 0} 2x + h = 2x$$

Notations for the Derivative

Let $y = f(x)$ be a differentiable function.

	Derivative Function	**Derivative at $x = a$**
Lagrange	$y'(x) = f'(x)$	$f'(a)$
Leibniz	$\dfrac{d}{dx}(f(x)) = \dfrac{dy}{dx} = \dfrac{df}{dx}$	$\dfrac{d}{dx}(f(x))\Big\vert_{x=a} = \dfrac{dy}{dx}\Big\vert_{x=a}$

Higher Derivatives:

- $f''(x), f'''(x), \ldots, f^{(n)}(x), \ldots$
- $\frac{d}{dx}\left(\frac{d}{dx}(f(x))\right) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ $\qquad \left(\frac{d}{dx}\right)^n (f(x)) = \frac{d^ny}{dx^n}$

Derivatives and Motion

A particle moves along a straight line. The function $s(t)$ represents its distance from a fixed point at time t.

- The **velocity** of the object at time t is $v(t) = s'(t) = \dfrac{ds}{dt}$
- The **acceleration** at time t is $a(t)$:

$$a(t) = s''(t) = v'(t) = \frac{d^2s}{dt^2} = \frac{dv}{dt}$$

- The **jerk** at time t is $j(t)$:

$$j(t) = s'''(t) = v''(t) = a'(t) = \frac{d^3s}{dt^3} = \frac{d^2v}{dt^2} = \frac{da}{dt}$$

Units

$$s(t) \text{ distance} \quad , \quad v(t)\,\frac{\text{distance}}{\text{time}} \quad , \quad a(t)\,\frac{\text{distance}}{\text{time}^2} \quad , \quad j(t)\,\frac{\text{distance}}{\text{time}^3}$$

Derivative Rules

Suppose that f and g are differentiable functions and c is a constant.

Derivative of Constants

$$\frac{d}{dx}(c) = 0$$

Constant Multiple Rule

$$\frac{d}{dx}(cf(x)) = c\,\frac{d}{dx}(f(x))$$

Sum and Difference Rules

$$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$$

The Power Rule

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Differentiating polynomials is now simple! $\quad \frac{d}{dx}\left(x^7 - 12x^4 + x^2 - x - 3\right)$

$$= \frac{d}{dx}\left(x^7\right) - 12\frac{d}{dx}\left(x^4\right) + \frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(x) + \frac{d}{dx}(-3) = 7x^6 - 48x^3 + 2x - 1$$

Exponential Functions and their Derivatives

Recall that an **exponential function** is a function of the form $f(x) = b^x$, where b is a positive constant. Can we determine their derivatives?

$$f'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x(b^h - 1)}{h} = b^x \boxed{\lim_{h \to 0} \frac{b^h - 1}{h}}$$

The derivative depends on the value of $\lim_{h \to 0} \frac{b^h - 1}{h}$ $\boldsymbol{= f'(0).}$

It will take some time, but we eventually will show that

$$\lim_{h \to 0} \frac{b^h - 1}{h} = \ln(b).$$

Exponential Functions and their Derivatives

Derivatives of Exponential Functions

For all positive real numbers b,

$$\frac{d}{dx}\left(b^x\right) \;=\; b^x\left(\lim_{h\to 0}\frac{b^h-1}{h}\right) \;=\; b^x\ln(b)$$

Euler's constant e is the unique number satisfying the equation

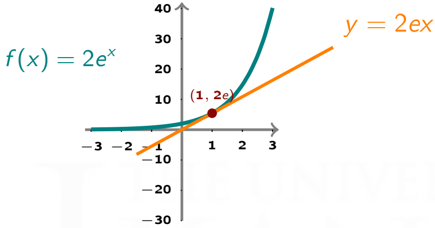
$$\ln(e) = \lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

(It so happens that $e \approx 2.718281845904523536028747135 27\ldots$)

Derivative of e^x

$$\frac{d}{dx}\left(e^x\right) = e^x$$

Example I: Find an equation of the tangent line to $f(x) = 2e^x$ at $x = 1$.



Solution: The tangent line passes through the point $(1, f(1))$ and has slope $f'(1)$.

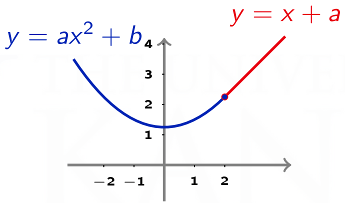
$$f(\overbrace{1}^{x_0}) = \underbrace{2e}_{y_0} \qquad f'(x) = 2e^x \qquad f'(1) = \underbrace{2e}_{\text{slope}}$$

Using point-slope form, the tangent line is

$$y - f(1) = f'(1)(x - 1)$$

$$\boxed{y - 2e = 2e(x - 1)} \text{ or } \boxed{y = (2e)x}$$

Example II: Let $f(x) = \begin{cases} ax^2 + b & x \le 2 \\ x + a & x > 2 \end{cases}$. Find the values for **a** and **b** which make the function differentiable.



Solutions: Note one equation is obtained using the continuity condition and another is obtained from slope of the tangent line from left and right.

Slopes:

$$\lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2}$$

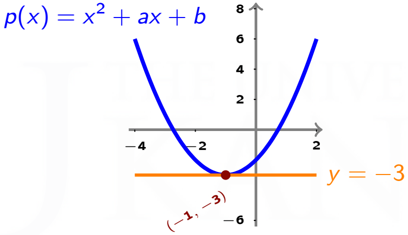
$$4a = 1 \implies \boxed{a = 0.25}$$

Continuity:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$$

$$4a + b = 2 + a \implies \boxed{b = 1.25}$$

Example III: Let $p(x) = x^2 + ax + b$. Find values for **a** and **b** which make the graph of $p(x)$ pass through $(-1, -3)$ and make the tangent line to p at $x = -1$ horizontal.



Solution: Note that $p'(x) = 2x + a$. A horizontal tangent line has slope 0, so $p'(-1) = 0$. We obtain two equations:

$$p(-1) = -3 \qquad p'(-1) = 0$$
$$1 - a + b = -3 \qquad -2 + a = 0$$

Solving this system gives $\boxed{a = 2 \text{ and } b = -2.}$